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The use of field theoretic methods for the study of flow in a heterogeneous porous medium

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Abstract. The effects of heterogeneities on the steady state flow of a single fluid in a porous medium are examined. It is argued that incomplete knowledge of the permeability requires the use of a stochastic model of the system. It is shown that the problem may be written as a field theory which allows a perturbation series to be expressed by diagrammatic means. This allows the calculation of effective permeability, the mean pressure and the pressure variance. The method, as well as recovering familiar results, gives a formal means of improving the approximation and approaching more complex systems.

1. Introduction

It is well known that disorder is equivalent to a field and the methods of field theory provide a language for estimating, say, the conductivity of a disordered alloy. Many similar applications have been made of diagrammatic series and this paper is another such application to a problem of growing technical importance.

This paper is concerned with the problem of correctly averaging the flow of a single fluid through a heterogeneous porous medium. It is necessary to define what is meant by a heterogeneous medium. Quite clearly on a microscopic level the properties of a porous rock change very rapidly and randomly. The porous medium consists of connected void spaces (pores) and rock. The process of formation of such a material leads to a complicated network of randomly shaped and sized pores, through which the fluid must flow. At this level then, to determine the flow the equations of fluid flow (the Navier-Stokes equations) are solved in the void region of the rock. This is clearly a hopeless task: the pore structure is far too complicated for a solution to be obtained, even numerically. However, an empirical law governing such flow (for a Newtonian fluid at low flow rate) has been known for a long time. This is Darcy's law (Darcy 1856, Collins 1961, Dullien 1979, Scheidegger 1974) which states that the fluid velocity is proportional to the pressure gradient across the fluid

$$\boldsymbol{q} = \frac{-K}{\eta} \cdot \nabla p \tag{1.1}$$

where η is the viscosity of the fluid and K, the constant of proportionality, is the permeability tensor representing the drag on the fluid by the microscopic structure of the medium.

Darcy's law is an average, over some volume, of the microscopic equations of flow. It assumes that there is a small volume over which average rock and fluid properties are approximately constant. This defines a microscopic length scale (with a characteristic length of some tens of microns) in the terminology of Haldorsen and Lake (1982) and Claridge (1972). There are two further length scales to be considered. There may be very long range trends in property values arising from large geological structures. This is the megascopic scale (after Haldorsen and Lake) with a characteristic length of the order of kilometres. For reasons soon to be outlined variations on this length scale are ignored. We will be considering variations on an intermediate length scale (centimetres to tens of metres) with no underlying megascopic trends and use this as the definition of a heterogeneous medium.

The heterogeneity of the medium affects the flow of fluids in the medium. For example, it can affect the dispersion of dissolved chemicals or the interface between two immiscible phases in an oil reservoir. Therefore it is of great importance that a satisfactory model of flow in a heterogeneous medium exists. Use is made of field theory to analyse the problem. To highlight the technique we treat the simpler problem of single-phase steady state flow.

In this simplified problem of single-phase steady state flow Darcy's law is used in conjunction with the equation of continuity:

$$\nabla \cdot \boldsymbol{q} = 0 \tag{1.2}$$

to give an equation for the fluid pressure:

$$\nabla \cdot K \nabla p = 0. \tag{1.3}$$

Ideally there would be permeability values throughout space and the pressure equation (1.3) (along with boundary conditions) would be solved to give the deterministic solution to the problem. However, this is not usually the case, we only know the permeability at a few isolated points and have little or no information about values in between. To be in a position to solve for the pressure we must interpolate for values of the permeability everywhere. Clearly this interpolation is not unique. This leads us to a probabilistic approach to the problem. A probability weighting is associated with each possible interpolation which corresponds with a belief in the occurrence of a particular permeability distribution. This can be based on samples taken from the reservoir, knowledge of the geology of the reservoir, analogy with geologically similar reservoirs or purely theoretical models of the reservoir's heterogeneity. Solving for the pressure in each of these cases and averaging will give a mean pressure field (corresponding to the pressure that would occur in an equivalent homogeneous medium with an effective homogeneous permeability) with fluctuations around this. The averages alluded to here are ensemble averages over the probability distributions mentioned above. If it is assumed that there are no long-range trends in property these ensemble averages may be replaced by spatial averages, since the system is now statistically homogeneous.

The statistical distribution of the permeability can be inferred from core data (Collins 1961, Law 1944). This may then be used in a numerical model such as a Monte Carlo simulation of the system (Freeze 1975, Smith and Freeze 1979, Smith and Brown 1982). For very large systems, such as a petroleum reservoir, the number of realisations to be solved can become prohibitively large. However, it is possible that we will still need to use Monte Carlo techniques for the more complicated two-phase problem. For this problem we show that the flow may be represented as a field theory which is equivalent to a zero-state Potts model. This is done by treating the pressure equation as a stochastic differential equation (Beran 1968, Adomian 1963, 1970). This approach has been used by other authors, in particular the perturbation series adopted here (Gelhar 1974, Bakr *et al* 1978, Gutjahr *et al* 1978, Gutjahr and

Gelhar 1981, Mizell *et al* 1982, Dagan 1981, 1982). However, these authors have terminated the series at low order whereas we are able to include some higher-order terms exactly. That we reproduce previous results is significant, suggesting that earlier results are stable to higher order in perturbation theory. Furthermore the method used in this paper allows for further refinement of the approximations.

2. Perturbation formulation

The problem to be solved is the pressure equation (1.3) where the permeability is given by a probability distribution. Without loss of generality the permeability may be taken to be isotropic, since the permeability is a real symmetric tensor (Dullien 1979) and so may be diagonalised by using normal coordinates. These coordinates may be rescaled to ensure that the tensor is isotropic. This coordinate system is used from now on. This is true if the anisotropy is homogeneous; if it is not, then the coordinate transformation described above will vary with position and thus enter into the differential equation (1.3) adding to the complexity of the solution. The system is assumed to be homogeneously anisotropic.

Define the Green function for the pressure equation (1.3) by

$$\nabla_{\boldsymbol{r}} \cdot K(\boldsymbol{r}) \nabla_{\boldsymbol{r}} G(\boldsymbol{r}, \boldsymbol{r}') = \delta(\boldsymbol{r} - \boldsymbol{r}'). \tag{2.1}$$

Applying the Green theorem to (2.1) and the Neumann condition of constant flux ($q = -K\nabla\phi$ from Darcy's law) gives the pressure ($\phi(\boldsymbol{r})$) as

$$\phi(\boldsymbol{r}) = \boldsymbol{q} \cdot \int G(\boldsymbol{r}, \boldsymbol{r}')\, \mathrm{d}\boldsymbol{S}'. \tag{2.2}$$

Consider perturbations about a homogeneous medium of permeability K_0 for which the Green function is given by

$$K_0 \nabla_{\boldsymbol{r}}^2 G_0(\boldsymbol{r}, \boldsymbol{r}') = \delta(\boldsymbol{r} - \boldsymbol{r}'). \tag{2.3}$$

Write the permeability as

$$K(\boldsymbol{r}) = K_0 + y(\boldsymbol{r}) \tag{2.4}$$

where $y(\boldsymbol{r})$ is the perturbation. With this the Green function is given by

$$K_0 \nabla^2 G = \delta(\boldsymbol{r} - \boldsymbol{r}') - \nabla \cdot y \nabla G. \tag{2.5}$$

Now the bare Green function G_0 is the inverse of the operator $K_0\nabla^2$ on the left-hand side of equation (2.6) so that this differential equation may be written as an integral equation:

$$G(\boldsymbol{r}, \boldsymbol{r}') = G_0(\boldsymbol{r}, \boldsymbol{r}') - \int G_0(\boldsymbol{r}, \boldsymbol{r}'') K_0 \nabla_{\boldsymbol{r}''} \cdot y(\boldsymbol{r}'') \nabla_{\boldsymbol{r}''} G(\boldsymbol{r}'', \boldsymbol{r}')\, \mathrm{d}\boldsymbol{r}'' \tag{2.6}$$

which may be written in Fourier transform:

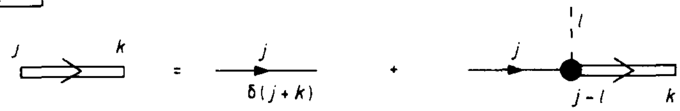
$$G(\boldsymbol{j}, \boldsymbol{k}) = G_0(\boldsymbol{j}) \delta(\boldsymbol{j} + \boldsymbol{k}) + G_0(\boldsymbol{j}) \int \mathrm{d}\boldsymbol{l}\, \mathrm{d}\boldsymbol{m}\, M(\boldsymbol{j}; \boldsymbol{l}, \boldsymbol{m}) y(\boldsymbol{l}) G(\boldsymbol{m}, \boldsymbol{k}) \tag{2.7}$$

where

$$M(\boldsymbol{j}: \boldsymbol{l}, \boldsymbol{m}) = K_0[(\boldsymbol{l} + \boldsymbol{m}) \cdot \boldsymbol{m}] \delta(\boldsymbol{l} + \boldsymbol{m} - \boldsymbol{j}). \tag{2.8}$$

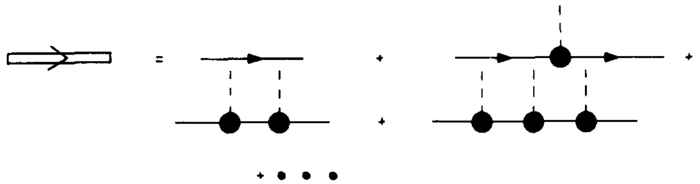
This provides the iterative scheme required.

This expansion is written using a diagrammatic representation. Use an arrow $\overset{k}{\rightarrow}$ to represent $G_0(\boldsymbol{k})$; a dot for M and a broken line - - - for y. Then if a thick arrow j ⇒ k is used for $G(\boldsymbol{j}, \boldsymbol{k})$ the integral equation (2.7) may be written as



(2.9)

This leads to the perturbation series



(2.10)

Hence the Green function may be developed as a perturbation expansion (2.10). This could be terminated at any order as has been done by previous authors (Gelhar 1974, Gutjahr and Gelhar 1981). However, we are often more interested in average properties and we can now show how averaging the perturbation series brings about a simplification which allows a partial summation of the series.

The perturbation expansion (2.10) is averaged term by term. The nth order term consists of the product of $n+1$, G_0; n, M and n, y. It is the permeability which has a probability distribution and so to determine the nth term in the series requires the nth moment of the y.

A log normal distribution of permeability is assumed (Law 1944)

$$P[K(\boldsymbol{r})] \sim \exp\left(-\frac{1}{2}\int\int \mathrm{d}\boldsymbol{r}\,\mathrm{d}\boldsymbol{r}' \ln\frac{K(\boldsymbol{r})}{K_g}\rho^{-1}(\boldsymbol{r}-\boldsymbol{r}')\ln\frac{K(\boldsymbol{r}')}{K_g}\right). \qquad (2.11)$$

This does not allow for simple evaluation of the moments. However, first assume $y(\boldsymbol{r})$ has zero mean which, from (2.4), means choosing K_0 such that it is equal to the arithmetic mean of the log normal distribution (2.11). This average is performed in appendix 1 to give

$$K_0 = K_g \exp(\tfrac{1}{2}\rho(0)). \qquad (2.12)$$

In a similar fashion (appendix 1) the higher moments of the permeability are found (A1.6). If the variance is small compared with the mean, a Gaussian approximation may be used for the moments of the permeability fluctuations. With this Gaussian approximation, the nth moment of the y is

$$\left\langle \prod_{i=1}^{n} y(\boldsymbol{k}_i) \right\rangle = 0 \qquad n \text{ odd}$$

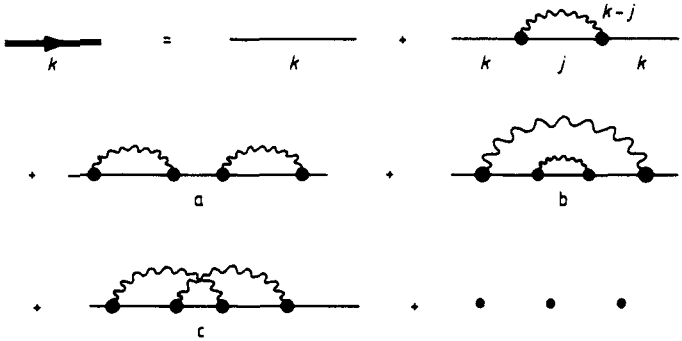
$$= \rho(\boldsymbol{k}_i)\delta(\boldsymbol{k}_1-\boldsymbol{k}_2)\rho(\boldsymbol{k}_3)\delta(\boldsymbol{k}_3-\boldsymbol{k}_4)+\rho(\boldsymbol{k}_1)\delta(\boldsymbol{k}_1-\boldsymbol{k}_3)\rho(\boldsymbol{k}_2)\delta(\boldsymbol{k}_2-\boldsymbol{k}_3) \qquad (2.13)$$

plus all other ways of pairing the $k \cdot n$ even.

In the diagrammatic representation the effect of averaging is to give zero if there is an odd number of y lines (the broken lines) and to 'tie' together the ends of pairs of broken lines in all possible ways to give a contribution $\rho(\boldsymbol{k})$ for each pair of lines tied together and write them as a wavy line. The delta functions in (2.13) ensure a conservation of wavenumber at each vertex. This is a consequence of the translational invariance of the system because the medium is assumed to be statistically homogeneous.

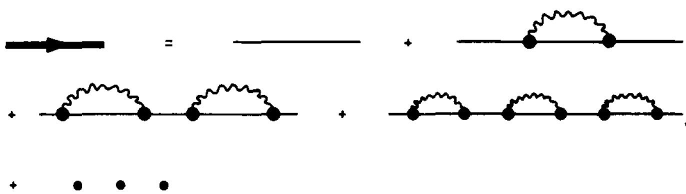
The perturbation expansion for the average Green function (represented by a full line ) is now developed. This is done with the help of the diagrammatic representation.

Averaging the expansion for the Green function (2.10) gives



(2.14)

At this point the series could be terminated at any arbitrary point on the assumption, for example, that the correlation between distant points (small wavenumber) is small. However, higher-order terms can be included by summing up parts of the series. Only diagrams which are made up of repeated parts are retained (type a in (2.14)). This is a reasonable approximation because the other diagrams (types b and c) contribute $1/k$ compared to diagrams of type a, but the correlation functions are small for low wavenumber. With this assumption the series for the average Green function becomes



(2.15)

This series may be summed by noting that it is a geometric progression

(2.16a)

which is equivalent to

$$\langle G(\boldsymbol{k})\rangle^{-1} = G_0^{-1}(\boldsymbol{k}) - \Sigma(\boldsymbol{k}) \tag{2.16b}$$

where $\Sigma(\boldsymbol{k})$ corresponds to  which is called the 'self-energy' by analogy with solid state and particle physics.

It is

$$\Sigma(\boldsymbol{k}) = K_0^2 \int \mathrm{d}\boldsymbol{j}(\boldsymbol{k}\cdot\boldsymbol{j})^2 \rho(\boldsymbol{k}-\boldsymbol{j}) G_0(\boldsymbol{j}). \tag{2.17}$$

The absence of polarisation diagrams () from this perturbation expansion indicates that the field theory is equivalent to a zero-state Potts model (an n-component vector field in the limit $n \to 0$).

3. The effective permeability

It is plausible that the mean behaviour of a heterogeneous medium is that of a homogeneous one but with an effective homogeneous permeability. If this were so then the average Green function would be given by (in analogy to the bare Green function)

$$\langle G(\boldsymbol{k})\rangle = -\frac{1}{K_{\text{eff}}k^2}. \tag{3.1}$$

In which case (2.16*b*) becomes

$$K_{\text{eff}} = K_0 + \frac{\Sigma(\boldsymbol{k})}{k^2}. \tag{3.2}$$

So the self-energy gives the effective permeability: in field theoretical terms it renormalises the permeability.

There will be further corrections to (3.2) arising from the other terms in the series (2.15) which have been ignored. These give rise to a renormalisation of the vertex function $M(\boldsymbol{k}: \boldsymbol{j}, \boldsymbol{l})$.

To calculate the effective permeability first substitute $\boldsymbol{l} = \boldsymbol{k} - \boldsymbol{j}$ and let the angle between $\boldsymbol{k}$ and $\boldsymbol{j}$ be θ. Then the self-energy is

$$\Sigma(\boldsymbol{k}) = K_0 k^2 \int \mathrm{d}\boldsymbol{l}\, \rho(\boldsymbol{l}) \cos^2\theta. \tag{3.3}$$

For an isotropic medium the correlation function does not depend on θ (an anisotropic correlation will be considered in appendix 2) and the angular part of the integral may be done directly to give (in d dimensions)

$$\Sigma(\boldsymbol{k}) = \frac{K_0 k^2 \pi^{d/2}}{\Gamma(1+d/2)} \int_0^\infty l^{d-1} \rho(l)\, \mathrm{d}l. \tag{3.4}$$

Now this integral may be done by writing the variance in Fourier transform:

$$\rho(r=0) = \int \rho(\boldsymbol{l})\, \mathrm{d}\boldsymbol{l}$$
$$= \frac{2\pi^{d/2}}{\Gamma(d/2)} \int_0^\infty l^{d-1}\rho(l)\, \mathrm{d}l. \tag{3.5}$$

Thus the self-energy may be written in terms of the permeability variance and from (3.2) the effective permeability can be written as

$$K_{\text{eff}} = K_0(1 - \rho(0)/d). \tag{3.6}$$

Now we use the argument that $\rho(0)$ must be small to allow us to use the Gaussian approximation, in which case the expression in brackets in (3.6) is the first-order approximation to an exponential

$$K_{\text{eff}} = K_0 \exp(-\rho(0)/d)$$
$$= K_g \exp[\rho(0)(1/2 - 1/d)]. \tag{3.7}$$

This result is not new (Gutjahr *et al* 1978). However, in Gutjahr *et al*'s derivation the flow was assumed to be essentially one dimensional. This assumption is not required here. They also terminated the perturbation series at second order. That this result is stable to higher orders, as shown here, is significant.

This result has several interesting and important limits. In one dimension it is known that the harmonic average is the correct effective permeability to use. Using (3.7) it can be seen that the effective permeability is $K_g \exp(-\rho(0)/2)$, which is indeed the harmonic average of the log normal distribution. Also it has long been thought, with considerable numerical corroboration (Warren and Price 1961), that for higher dimensions the geometric mean is the correct effective permeability. From (3.7) we see that in two dimensions the effective permeability is K_g, which is the geometric mean. For three dimensions it is $K_g \exp(\rho(0)/6)$ which is very close to the geometric mean (recall that $\rho(0)$ is small). This discrepancy may be because there are higher-order terms still missing from the perturbation series, or because the geometric average is not quite exact for three dimensions. This behaviour is also found if a mean field (or effective medium) theory is used (Dagan 1979, Koplik 1982). Also the geometric mean has been shown to be exact for the log normal distribution in two dimensions (Matheron 1967). As an example of the use of this result comparison was made with Warren and Price's (1961) published data. They used a three-dimensional sandpack with a heterogeneous permeability distribution with the following parameters: arithmetic mean, 70.2 Darcies; geometric mean, 47 Darcies; harmonic mean, 29.5 Darcies; variance in log permeability ~0.86. Equation (3.7) gives the effective permeability as 54.2 Darcies. In fact the measured effective permeability was 60.5 Darcies. The discrepancy is probably because the permeability fluctuations are quite large. In any case this estimate is in better agreement than their value of about 46 Darcies.

This effective permeability can be used to examine the mean fluid pressure in the medium. From (2.2) the mean pressure can be written as

$$\langle \phi(\boldsymbol{r}) \rangle = \boldsymbol{q} \cdot \int \langle G(\boldsymbol{r}, \boldsymbol{r}') \rangle \, \mathrm{d}\boldsymbol{S}'. \tag{3.8}$$

The renormalised Green function is similar to the bare Green function but with the effective, instead of the bare, permeability. Hence

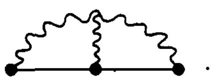
$$\langle \phi(\boldsymbol{r}) \rangle = \frac{\boldsymbol{q}}{K_{\mathrm{eff}}} \cdot \int |\boldsymbol{r} - \boldsymbol{r}'|^{2-d} \, \mathrm{d}\boldsymbol{S}'. \tag{3.9}$$

(Note that in two dimensions the Green function is $\ln|\boldsymbol{r} - \boldsymbol{r}'|$.)

An important conclusion of this work is that the mean pressure and the effective permeability do not depend on the correlation length, for the isotropic nearest-neighbour model. The anisotropic model is considered in appendix 2.

It is worthwhile recalling the assumptions made in arriving at these results. First it has been assumed that the problem is amenable to solution by perturbation theory. That is, the Green function must be analytic around the bare Green function and so has a Taylor series expansion in the sense of (2.10). In general, perturbation theory is valid if the perturbed state is qualitatively similar to the unperturbed state which it will be if the system is not close to a percolation threshold. For the continuous distribution used there is no percolation threshold. However, if a permeability distribution with a finite fraction of zero permeability had been used then the above approach may not be valid.

Next we have made an assumption that a sharply peaked log normal distribution may be replaced by a Gaussian distribution. If the log normal distribution is retained then three- (and higher-) point correlation functions will be present in the perturbation series (2.14), represented by diagrams like



As the covariance decreases these terms become less important and the series reduced to the one has been used.

In summing the diagrammatic series only a certain class of terms have been retained. This is based on the assumption that the correlation dies away quickly for large distances (the correlation length is small, but not zero). This assumption could be corrected by inclusion of the other terms by vertex renormalisation. Finally we have written the effective permeability in the exponential form (3.7). The only real justification for this is that it gives the right limits in one and two dimensions.

4. The pressure variance

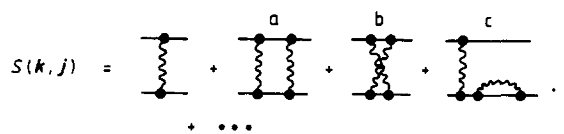
As stressed in the introduction the paucity of accurate data leads to a probabilistic solution to the problem. The mean values that were determined in the previous section give an estimate of the pressure that would be expected. A further useful parameter to study is the pressure variance defined as

$$\sigma_\phi^2(\boldsymbol{r}_1, \boldsymbol{r}_2) = \langle\phi(\boldsymbol{r}_1)\phi(\boldsymbol{r}_2)\rangle - \langle\phi(\boldsymbol{r}_1)\rangle\langle\phi(\boldsymbol{r}_2)\rangle \tag{4.1}$$

which can be written in terms of the Green functions as

$$\begin{aligned}\sigma_\phi^2(\boldsymbol{r}_1\boldsymbol{r}_2) &= \boldsymbol{qq}{:}\int \mathrm{d}\boldsymbol{S}_1'\,\mathrm{d}\boldsymbol{S}_2'\{\langle G(\boldsymbol{r}_1, \boldsymbol{r}_1')G(\boldsymbol{r}_2, \boldsymbol{r}_2')\rangle - \langle G(\boldsymbol{r}_1, \boldsymbol{r}_1')\rangle\langle G(\boldsymbol{r}_2, \boldsymbol{r}_2')\rangle\}\\ &= \boldsymbol{qq}{:}\int \mathrm{d}\boldsymbol{S}_1'\,\mathrm{d}\boldsymbol{S}_2'\int \mathrm{d}\boldsymbol{k}\,\mathrm{d}\boldsymbol{j}\exp[\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{r}_1-\boldsymbol{r}_1')+\mathrm{i}\boldsymbol{j}\cdot(\boldsymbol{r}_2-\boldsymbol{r}_2')]\\ &\quad\times\{\langle G(\boldsymbol{k})G(\boldsymbol{j})\rangle - \langle G(\boldsymbol{k})\rangle\langle G(\boldsymbol{j})\rangle\}.\end{aligned} \tag{4.2}$$

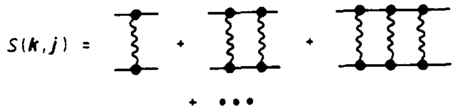
The perturbation series derived in § 2 is used again and also the diagrammatic expansion of equation (2.10). If we denote the term in the braces in (4.2) by $S(k, j)$ then this is found by the following expansion:



(4.3)

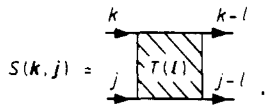
Note that there is a cancellation of the average (renormalised) propagators.

Again a partial summation of this series can be achieved by selective retention of certain terms. Terms of type c are accounted for by using renormalised propagators rather than bare ones for the straight lines. Terms of type b are an order of wavenumber smaller than those of type a and so are only significant for low wavenumber. However, at low wavenumber the correlation function becomes very small if the correlation length is small. Hence we only retain terms of type a:



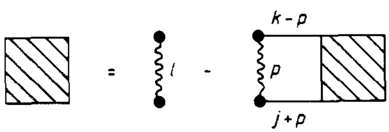
(4.4)

where the straight lines represent the renormalised propagator. Represent $S(\boldsymbol{k}, \boldsymbol{j})$ by using a square box:



(4.5)

Then this term $T(\boldsymbol{l})$ may be written in a Dyson equation form as



(4.6)

which is the diagrammatic representation of the following integral equation

$$T(\boldsymbol{l}) = K_0^2(\boldsymbol{k}\cdot\boldsymbol{l})(\boldsymbol{j}\cdot\boldsymbol{l})\rho(\boldsymbol{l}) - K_0^2 \int \mathrm{d}\boldsymbol{p}\, \rho(\boldsymbol{p})(\boldsymbol{k}\cdot\boldsymbol{p})(\boldsymbol{j}\cdot\boldsymbol{p})G(\boldsymbol{k}-\boldsymbol{p})G(\boldsymbol{j}+\boldsymbol{p})T(\boldsymbol{p}-\boldsymbol{l}). \tag{4.7}$$

Now, it has been assumed that the system is statistically homogeneous so the pressure variance can only depend on $\boldsymbol{r}_1 - \boldsymbol{r}_2$. Hence only the $\boldsymbol{j} = -\boldsymbol{k}$ term is important. This means that the integral equation (4.7) reduces to

$$T(\boldsymbol{l}) = K_0^2(\boldsymbol{k}\cdot\boldsymbol{l})^2\rho(\boldsymbol{l}) + \frac{K_0^2}{K_{\mathrm{eff}}^2}\int \frac{\mathrm{d}\boldsymbol{p}\,\rho(\boldsymbol{p})(\boldsymbol{k}\cdot\boldsymbol{p})^2}{[(\boldsymbol{k}-\boldsymbol{p})^2]^2}T(\boldsymbol{p}-\boldsymbol{l}). \tag{4.8}$$

Unfortunately this integral equation is not amenable to a simple solution. However, we can take it to first order and use the approximation

$$T(\boldsymbol{l}) = K_0^2(\boldsymbol{k}\cdot\boldsymbol{l})^2\rho(\boldsymbol{l}) \tag{4.9}$$

which may be used in conjunction with the defining equation (4.5) to give

$$S(\boldsymbol{k}, -\boldsymbol{k}) = K_0^2 G(\boldsymbol{k})G(-\boldsymbol{k}) \int \mathrm{d}\boldsymbol{l}\, (\boldsymbol{k}\cdot\boldsymbol{l})^2\rho(\boldsymbol{l})[G(\boldsymbol{k}-\boldsymbol{l})]^2. \tag{4.10}$$

This integral may be done by writing the angle between $\boldsymbol{k}$ and $\boldsymbol{l}$ as θ. Then

$$S(\boldsymbol{k}, -\boldsymbol{k}) = \frac{K_0^2}{k_{\mathrm{eff}}^4 k^2}\int \frac{\mathrm{d}\boldsymbol{l}\, l^2\cos^2\theta\rho(l)}{(k^2+l^2-2kl\cos\theta)^2}. \tag{4.11}$$

At large distances the low k behaviour of this integral dominates. Hence

$$\lim_{k\to 0} S(\boldsymbol{k}, -\boldsymbol{k}) = \frac{1}{k^2}\frac{K_0^2}{K_{\mathrm{eff}}^4}\int \frac{\mathrm{d}\boldsymbol{l}\,\rho(l)}{l^2}\cos^2\theta. \tag{4.12}$$

For an isotropic correlation function the same trick of separating the angular part of the integral may be used to write this as

$$\lim_{k\to 0} S(\boldsymbol{k}, -\boldsymbol{k}) = \frac{1}{k^2}\frac{K_0^2}{dK_{\text{eff}}^4}\int \frac{dl\,\rho(l)}{l^2}. \tag{4.13}$$

This integral can be written back in real space by inverting the Fourier transform which would result in the double integral of the real space correlation function

$$\int \frac{dl\,\rho(l)}{l^2} = \int_0^R \int_0^r \rho(r')\,dr'\,dr'\bigg|_{R=0}. \tag{4.14}$$

For typical correlation functions with exponential decay over some correlation length λ this is equivalent to the square of this length. Hence, finally, the pressure covariance may be written as the following surface integral:

$$\sigma_\phi^2(\boldsymbol{r}_1 - \boldsymbol{r}_2) = \frac{q^2\rho(0)\lambda^2\exp(2\rho(0)/d)}{dK_{\text{eff}}^2}\int d\boldsymbol{S}'\,d\boldsymbol{S}\,|r-r'|^{2-d}. \tag{4.15}$$

This result depends on the shape of the boundary through which this flow takes place. If this is written as some general geometric factor S_d this may be written in the general form

$$\sigma_\phi^2 = \frac{q^2\exp(2\rho(0)/d)}{dK_{\text{eff}}^2}\rho(0)\lambda^2 S_d. \tag{4.16}$$

This is in agreement with previous authors where the geometric factor S_d for specific cases may be found (Bakr *et al* 1978, Gutjahr *et al* 1978, Gutjahr and Gelhar 1981, Mizell *et al* 1982, Dagan 1981, 1982).

5. Summary

A new method of analysing the problem of single-phase flow in a heterogeneous porous medium has been developed. It has been shown that the problem is equivalent to a field theory. The nature of the field is that of an n-component vector field in the limit as $n \to 0$ (a zero-state Potts model). This has allowed the application of field theoretic techniques to an old problem. In particular, diagrammatic methods have been used to sum parts of an infinite perturbation series. This method has been used to recover familiar results for the effective permeability of a heterogeneous medium. This was done by using a self-energy renormalisation. The assumption of essentially linear flow used by previous authors was not required here. Also higher-order terms were included in this evaluation of effective permeability.

The method was also applied to the calculation of pressure covariance. This essentially gives a measure of the error involved in replacing a heterogeneous one with the effective permeability. Here again familiar results have been recovered.

By writing the problem within the field theoretical framework it should be possible to solve more general problems than that considered here. For example, non-perturbative methods may be used when the permeability fluctuations are small. The purpose of this paper was to demonstrate the field theoretic nature of the problem and how these techniques can be used to recover familiar results.

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Appendix 1

The problem is to find the nth moment of the log normal distribution

$$\mu_n = \left\langle \prod_{i=1}^{n} K(\boldsymbol{r}_i) \right\rangle$$
$$= \int \mathrm{D}\ln K \prod_{i=1}^{n} K(\boldsymbol{r}_i) \exp\left(-\tfrac{1}{2}\int\int \mathrm{d}\boldsymbol{r}\,\mathrm{d}\boldsymbol{r}' \ln\frac{K(\boldsymbol{r})}{K_g}\rho^{-1}(\boldsymbol{r}-\boldsymbol{r}')\ln\frac{K(\boldsymbol{r}')}{K_g}\right)$$
$$\times\left[\int \mathrm{D}\ln K \exp\left(-\tfrac{1}{2}\int\int \mathrm{d}\boldsymbol{r}\,\mathrm{d}\boldsymbol{r}' \ln\frac{K(\boldsymbol{r})}{K_g}\rho^{-1}(\boldsymbol{r}-\boldsymbol{r}')\ln\frac{K(\boldsymbol{r}')}{K_g}\right)\right]^{-1}. \quad \text{(A1.1)}$$

Changing variables to $K(\boldsymbol{r}) = K_g\,\mathrm{e}^{u(\boldsymbol{r})}$ gives

$$\mu_n = K_g^n \int \mathrm{D}u(\boldsymbol{r}) \exp\left(\sum_{i=1}^{n} u(\boldsymbol{r}_i) - \tfrac{1}{2}\int\int \mathrm{d}\boldsymbol{r}\,\mathrm{d}\boldsymbol{r}'\, u(\boldsymbol{r})\rho^{-1}(\boldsymbol{r}-\boldsymbol{r}')u(\boldsymbol{r}')\right)$$
$$\times\left[\int \mathrm{D}u(\boldsymbol{r}) \exp\left(-\tfrac{1}{2}\int\int \mathrm{d}\boldsymbol{r}\,\mathrm{d}\boldsymbol{r}'\, u(\boldsymbol{r})\rho^{-1}(\boldsymbol{r}-\boldsymbol{r}')u(\boldsymbol{r}')\right)\right]^{-1}. \quad \text{(A1.2)}$$

At this point it is useful to introduce the Fourier variables:

$$u(\boldsymbol{r}) = \int \exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r})u(\boldsymbol{k})\,\mathrm{d}\boldsymbol{k} \quad \text{(A1.3)}$$

so that

$$\mu_n = K_g^n \int \mathrm{D}u(\boldsymbol{k}) \exp\left(\sum_i \int \exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}_i)\,\mathrm{d}\boldsymbol{k}\, u(\boldsymbol{k}) - \tfrac{1}{2}\int \mathrm{d}\boldsymbol{k}\, u(\boldsymbol{k})\rho^{-1}(\boldsymbol{k})u(-\boldsymbol{k})\right)$$
$$\times\left[\int \mathrm{D}u(\boldsymbol{k}) \exp\left(-\tfrac{1}{2}\int \mathrm{d}\boldsymbol{k}\, u(\boldsymbol{k})\rho^{-1}(\boldsymbol{k})u(-\boldsymbol{k})\right)\right]^{-1}. \quad \text{(A1.4)}$$

The exponential in the numerator can be rewritten by completing the square

$$\int \mathrm{d}\boldsymbol{k}\, u(\boldsymbol{k})\rho^{-1}(\boldsymbol{k})u(-\boldsymbol{k}) - 2\sum_{i=1}^{n} \exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}_i)\mathrm{i}u(\boldsymbol{k})$$
$$= \int \mathrm{d}\boldsymbol{k}\left[\left(u(\boldsymbol{k}) - \rho(\boldsymbol{k})\sum_{i=1}^{n}\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}_i)\right)\rho^{-1}(\boldsymbol{k})\right.$$
$$\left.\times\left(u(-\boldsymbol{k}) - \rho(\boldsymbol{k})\sum_{i=1}^{n}\exp(-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}_i)\right)\right]$$
$$-\sum_{i,j=1}^{n}\int \mathrm{d}\boldsymbol{k}\,\rho(\boldsymbol{k})\exp[\mathrm{i}\boldsymbol{k}\cdot(\boldsymbol{r}_i-\boldsymbol{r}_j)]. \quad \text{(A1.5)}$$

So that if new Fourier variables $\tilde{u}(\boldsymbol{k}) = u(\boldsymbol{k}) - \rho(\boldsymbol{k})\Sigma_{i=1}^{n} \exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r}_i)$ are introduced into (A1.4) the Gaussian integral may be performed to give

$$\mu_n = K_g^n \exp\left(\tfrac{1}{2} \sum_{i,j=1}^{n} \rho(\boldsymbol{r}_i - \boldsymbol{r}_j)\right). \tag{A1.6}$$

Explicitly the first few moments are

$$\mu_1 = K_g \exp(\tfrac{1}{2}\rho(0)) \tag{A1.7}$$

$$\mu_2 = \langle K(\boldsymbol{r}_1)K(\boldsymbol{r}_2)\rangle = K_g^2 \exp(\rho(0) + \rho(\boldsymbol{r}_1 - \boldsymbol{r}_2)) \tag{A1.8}$$

$$\mu_3 = \langle K(\boldsymbol{r}_1)K(\boldsymbol{r}_2)K(\boldsymbol{r}_3)\rangle = K_g^3 \exp[\tfrac{3}{2}\rho(0) + \rho(\boldsymbol{r}_1 - \boldsymbol{r}_2) + \rho(\boldsymbol{r}_1 - \boldsymbol{r}_3) + \rho(\boldsymbol{r}_2 - \boldsymbol{r}_3)]. \tag{A1.9}$$

Appendix 2

For an anisotropic medium a suitable correlation function for the permeability is

$$\rho(\boldsymbol{k}) = \frac{\rho(0)\Pi_{i=1}^{d} \lambda_i}{\pi^{d/2}\Gamma((4-d)/2)[\Sigma_{i=1}^{d} (\lambda_i k_i)^2 + 1]^2} \tag{A2.1}$$

where λ_i are the correlation lengths in the principal coordinate directions.

Using this the 'self-energy' (2.17) becomes

$$\Sigma(\boldsymbol{k}) = \frac{-K_0\rho(0)\Pi_i \lambda_i}{\pi^{d/2}\Gamma((4-d)/2)} \int \frac{\mathrm{d}\boldsymbol{j}[\boldsymbol{k}\cdot(\boldsymbol{k}-\boldsymbol{j})]^2}{[(\boldsymbol{\lambda}\cdot\boldsymbol{j})^2+1](\boldsymbol{k}-\boldsymbol{j})^2} \tag{A2.2}$$

where $\boldsymbol{\lambda}$ is the vector of principal axis correlation lengths. If θ is defined as the angle between $\boldsymbol{k}$ and $\boldsymbol{k}-\boldsymbol{j}$ and ϕ is the angle between $\boldsymbol{\lambda}$ and $\boldsymbol{j}$ then the 'self-energy' becomes

$$\begin{aligned}\Sigma(\boldsymbol{k}) &= \frac{-K_0\rho(0)\Pi_i \lambda_i k^2}{\pi^{d/2}\Gamma((4-d)/2)} \int \frac{j^{d-1}\,\mathrm{d}j\,\mathrm{d}\Omega\cos^2\theta}{(\lambda^2 j^2\cos^2\phi + 1)^2} \\ &= \frac{K_0\rho(0)\Gamma(d/2)k^2}{2\pi^{d/2}\lambda^d} \prod_i \lambda_i \int \frac{\mathrm{d}\Omega\cos^2\theta}{\cos^d\phi}.\end{aligned} \tag{A2.3}$$

We cannot do the angular integral directly but it must be independent of whether the correlation is anisotropic.

For the isotropic case (3.5) gives

$$\Sigma(\boldsymbol{k}) = \frac{-k^2K_0}{d}\rho(0) \tag{A2.4}$$

whereas (A2.3) would give

$$\Sigma(\boldsymbol{k}) = \frac{-k^2K_0\rho(0)}{2\pi^{d/2}} \frac{\Gamma(d/2)}{d^{d/2}} \int \frac{\mathrm{d}\Omega\cos^2\theta}{\cos^d\phi} \tag{A2.5}$$

from which we deduce that

$$\int \frac{\mathrm{d}\Omega\cos^2\theta}{\cos^d\phi} = \frac{2\pi^{d/2}d^{d/2-1}}{\Gamma(d/2)}. \tag{A2.6}$$

Hence the self-energy is

$$\Sigma(\boldsymbol{k}) = \frac{-K_0\rho(0)k^2 d^{d/2-1}\Pi_{i=1}^{d}\lambda_i}{(\Sigma_{i=1}^{d}\lambda_1^2)^{d/2}}. \tag{A2.7}$$

This implies that the effective permeability is

$$K_{\text{eff}} = K_g \exp\left[\rho(0)\left(\frac{1}{2} - \frac{d^{d/2-1}\Pi_i \lambda_i}{(\Sigma_i \lambda_i^2)^{d/2}}\right)\right]. \tag{A2.8}$$

When the correlation is isotropic this reduces to the previous result (3.7).

If one or more of the correlation lengths becomes infinite (the medium is stratified) the effective permeability becomes (except in one dimension)

$$K_{\text{eff}} = K_g \exp(\rho(0)/2) \tag{A2.9}$$

which is the arithmetic mean of the permeabilities. It has long been known (Scheidegger 1974, Craig 1971) that the arithmetic mean is the correct average to use when the flow is parallel to the strata. Here we have removed the anisotropy from the permeability (in writing it as a scalar) and so assumed that the flow through the boundary is isotropic (and constant). The layering then has the effect of channelling the flow to be parallel to the strata and so the arithmetic mean is found for the effective permeability. This result (A2.8) is the same as that found previously by Gelhar and Axness (1983, equations (52)-(60)) expressed in more compact form.

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